

ON RIGIDITY OF GENERALIZED CONFORMAL STRUCTURES

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ABSTRACT. The classical Liouville Theorem on conformal transformations determines local conformal transformations on the Euclidean space of dimension ≥ 3 . Its natural adaptation to the general framework of Riemannian structures is the 2-rigidity of conformal transformations, that is such a transformation is fully determined by its 2-jet at any point. We prove here a similar rigidity for generalized conformal structures defined by giving a one parameter family of metrics (instead of scalar multiples of a given one) on each tangent space.

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1. INTRODUCTION

Rough notion. For a vector space E , let $\text{Sym}(E)$ be the space of symmetric bilinear forms on E , $\text{Sym}^+(E)$ those which are positive definite, and $\text{Sym}^*(E)$ the non-degenerate ones.

For a manifold M , one defines similarly fiber bundles $\text{Sym}(TM)$, $\text{Sym}^+(TM)$ and $\text{Sym}^*(TM)$ associated to its tangent bundle TM .

A Riemannian metric is nothing but a section of $\text{Sym}^+(TM)$. Recall on the other hand that a (Riemannian) conformal structure consists in giving a class $[g]$ of Riemannian metrics, for the conformal equivalence relation \sim between metrics: $g_1 \sim g_2$ if there exists a function σ on M such that $g_1 = e^\sigma g_2$. Thus, a conformal structure consists in giving a section of the projectivized of $\text{Sym}^+(TM)$.

Equivalently, a conformal structure consists in giving for each point $x \in M$, a half line in $\text{Sym}^+(T_x M)$.

We are now going to introduce a first rough definition of generalized conformal structures (GCS for **short**) by associating to each $x \in M$ a (non-parameterized) curve in $\text{Sym}^+(T_x M)$. Say, this consists in giving a subset $\mathcal{C} \subset \text{Sym}^+(TM)$ such

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that the fibers of the projection $\mathcal{C} \rightarrow M$ have dimension ≤ 1 (and are non-empty). One naturally defines the image of such a structure \mathcal{C} by a diffeomorphism, and by the way define an automorphism group $Aut(\mathcal{C})$. In the sequel, automorphisms will be alternatively called isometries.

Our goal is to study such objects from the point of view of being “rigid geometric structures”? Roughly speaking, d -rigidity means that an automorphism is fully determined by its jet up to order d at any point. We have here two “limit” cases, that where the \mathcal{C} -fibers are point (a Riemannian metric), and the other where the \mathcal{C} -fibers are half-lines (a conformal structure). It is known that Riemannian metrics are 1-rigid, whereas conformal structures are 2-rigid in dimension ≥ 3 ; this is the essence of classical Liouville Theorem. Our generalized case here when the \mathcal{C} -fibers are general curves may be expected to be of as rigid as the conformal case, that is one has 2-rigidity.

1.0.1. *A First example.* Let us start by this general example which will give evidence that some topological tameness hypotheses on \mathcal{C} are in order. Let ϕ^t be a flow on M and g_0 (any initial) metric on M . For any x , give \mathcal{C}_x as the (parameterized) curve $t \rightarrow (\phi_*^t g_0)_x \in \text{Sym}^+(T_x M)$ (here $\phi_*^t g_0$ is the image of g_0 by ϕ^t).

Observe that $\phi^t \in Aut(\mathcal{C})$. Thus, any flow gives rise to a rough GCS with a non-trivial automorphism group, which may have a strong dynamics. One can not expect for such structure to behave as a nice geometric structure!

1.0.2. *(Regular) Definition.* We are now going to propose a definition of GCS which will be proved to be adapted to our rigidity hope, just by assuming that the corresponding subset \mathcal{C} to be a submanifold.

More precisely, let us say \mathcal{C} is a (regular) GCS if \mathcal{C} is a submanifold of dimension $\dim M + 1$ in $\text{Sym}^+(TM)$, which is transverse to the fibers (of $\text{Sym}^+(TM) \rightarrow M$). Equivalently, the projection $\mathcal{C} \rightarrow M$ is a submersion and $\dim \mathcal{C} = \dim M + 1$.

Each fiber \mathcal{C}_x is thus a (non-necessarily connected) 1-dimensional submanifold.

In the case of a classical conformal structure, \mathcal{C} is in fact a closed submanifold and it fibers over M .

Let us say that \mathcal{C} is **generic** if the tangent direction of \mathcal{C}_x at any of its points belongs to $\text{Sym}^*(T_x M)$. In other words, if \mathcal{C}_x is parameterized as a curve $t \in \mathbb{R} \rightarrow c_x(t) \in \text{Sym}(T_x M)$, then $c'(t)$ is assumed to be non-degenerate. For example, classical conformal structures are generic.

1.0.3. *A second example, Infinitesimally Homogeneous case.* (see 3.1). Let us consider the situation where there is a (non-parameterized) curve $\mathcal{C}_0 \subset \text{Sym}^+(\mathbb{R}^n)$ (a 1-dimensional submanifold), such that for any x , $\mathcal{C}_x = A_x^*(\mathcal{C}_0)$, where $A_x : \mathbb{R}^n \rightarrow T_x M$ is a linear isomorphism (and A_x^* is the associated map $\text{Sym}^+(\mathbb{R}^n) \rightarrow \text{Sym}^+(T_x M)$).

If the dependence $x \rightarrow A_x$ is smooth (in a natural sense), then \mathcal{C} is a GCS, which as in the standard conformal case, gives rise to a fibration $\mathcal{C} \rightarrow M$.

Let us here mention one useful and beautiful property of this modular space $\text{Sym}^+(\mathbb{R}^n)$, or more generally any $\text{Sym}^+(E)$, for E a linear space; this is the space of “linear” Riemannian metrics on E , and it admits itself a canonical Riemannian metric, which makes it as (universal) symmetric space under the natural action of $\text{GL}(E)$ (see 3.2.1).

For \mathcal{C}_0 , consider H its stabilizer subgroup in $\text{GL}(n, \mathbb{R})$.

For any $x \in M$, consider I_x the set of isomorphisms $T_x M \rightarrow \mathbb{R}^n$ sending \mathcal{C}_x to \mathcal{C}_0 . This is clearly an H -orbit in the $\text{GL}(n, \mathbb{R})$ -space $\text{Isom}(T_x M, \mathbb{R}^n)$, that is the fiber over x of the frame bundle $P_M \rightarrow M$. When x runs over M , we therefore get a section of $P_M/H \rightarrow M$, that is an H -structure on M .

Conversely, an H -structure gives naturally a GCS of type \mathcal{C}_0 . Indeed, by definition of an H -structure, it consists in giving for any x , an H -orbit I_x as above. The pull back \mathcal{C}_x of the curve \mathcal{C}_0 by any element of I_x does not depend on the choice of such element.

1.0.4. Rigidity. Let ϕ be a diffeomorphism of M and ϕ^* its induced action on $\text{Sym}(TM)$. Then ϕ is an automorphism of \mathcal{C} (a GCS on M) if $\phi^*(\mathcal{C}) = \mathcal{C}$.

If $p \in M$, define ϕ to be isometric up to order 1 at p , if $\phi(p) = p$ and $(\phi^*(\mathcal{C}))_p = \mathcal{C}_p$, i.e. $\phi^*(\mathcal{C})$ and \mathcal{C} meet along \mathcal{C}_p . We say that ϕ is isometric up to order $d \geq 1$ (or simply a d -isometry) if $\phi^*(\mathcal{C})$ and \mathcal{C} have contact of order $(d-1)$ along \mathcal{C}_p . (We will say that ϕ has a trivial d -jet if it has the same d -jet as the identity at p).

Rigidity at order 2 of classical conformal structures in dimension ≥ 3 , is essentially equivalent to the classical Liouville Theorem stating that any (local) conformal transformation of an Euclidean space of dimension ≥ 3 , is a composition of a translation, a similarity and an inversion (see for instance [5, 13] and [8]). There are many approaches to this rigidity, including that by the theory of H -structures of finite type, via computation of the prolongation spaces for the conformal group $H = \mathbb{R}.\text{O}(n)$, see [12, 14, 10, 1]. Here we generalize to generic GCS:

Theorem 1.1 (Generalized Liouville Theorem). *Let \mathcal{C} be a generic generalized conformal structure on a manifold of dimension ≥ 3 .*

Then \mathcal{C} is 2-rigid, that is a 3-isometry with a trivial 2-jet, has a trivial 3-jet.

Example 1.2 (A non rigid example). *Consider canonical coordinates (x^1, \dots, x^n) on \mathbb{R}^n . Endow it with \mathcal{C} the (constant) GCS given by the curve of Euclidean metrics $t(dx^1)^2 + (dx^2)^2 + \dots + (dx^n)^2$ ($t > 0$). This \mathcal{C} is in fact an H -structure. Any diffeomorphism ϕ of the form $\phi(x^1, \dots, x^n) = (f(x^1), x^2, \dots, x^n)$ is isometric. This structure is not rigid, indeed \mathcal{C} is not generic.*

Note however that it may happen for a GCS to be rigid, even if it is not generic (such a situation is thus not covered by our result). For instance an H -structure with H a one parameter subgroup of $\text{GL}(n, \mathbb{R})$ has finite type iff the Lie subalgebra

of H contains no matrices of rank 1, in which case the structure has finite type 1, i.e. it is 1-rigid like a Riemannian metric (see §3.1).

Remark 1.3. More generalizations of conformal structures can be obtained by relaxing the dimension condition on \mathcal{C} , say by assuming $\dim \mathcal{C} = \dim M + l$, where l may be bigger than 1. The rigidity discussion will then depend on l and $\dim M$?

2. FURTHER INVESTIGATIONS

2.1. Interplay with Lightlike metrics. Our motivation behind the study of GC structures was in fact their relation with the lightlike ones that we considered in [3]. Recall that a **lightlike** metric g on a manifold \mathcal{V} is a tensor which is a positive non-definite quadratic form of 1-dimensional kernel in each tangent space of \mathcal{V} [2]. The kernel of g is a direction field N , tangent to a 1-dimensional foliation \mathcal{N} (called null or characteristic).

Let X be a non-singular vector field tangent to N . The lightlike structure is said to be **transversally Riemannian** if the Lie derivative $L_X g = 0$. Let us say g is **nowhere** transversally Riemannian, if $L_X g(x) \neq 0$, for any x . In the stronger situation where $L_X g$ is non-degenerate on $T\mathcal{V}/N$, g is said to be **generic**.

2.1.1. From GCS to lightlike structures. Let $\mathcal{C} \subset \text{Sym}^+(TM)$ be a GCS on M and $\pi : \mathcal{C} \rightarrow M$ the projection. Let $x \in M$, $q \in \mathcal{C}_x = \pi^{-1}(x)$, and consider the projection $d_q \pi : T_q \mathcal{C} \rightarrow T_x M$. Now, let q play the role of a (definite) scalar product on $T_x M$, its pull back by $d_q \pi$ is a lightlike scalar product on $T_q \mathcal{C}$. We get in this way a tautological lightlike metric on \mathcal{C} .

Observe that this lightlike metric on \mathcal{C} is nowhere transversally Riemannian. Observe also that \mathcal{C} is generic as a GCS iff its lightlike metric is generic (as defined previously).

2.1.2. Simple lightlike manifolds. We will call a lightlike manifold (\mathcal{V}, g) **simple** if it satisfies some global tameness conditions:

a) First, we will assume a topological regularity condition on the quotient space \mathcal{V}/\mathcal{N} : there is a (Hausdorff) manifold M and a submersion $\pi : \mathcal{V} \rightarrow M$, such that the connected components of its levels are the leaves of \mathcal{N} .

- Now, if one projects on $T_x M$ all the lightlike scalar products g_y on $T_x \mathcal{V}$, for y in the \mathcal{N} -leaf over x , then one gets a curve in $\text{Sym}^+(T_x M)$. This is a “rough GCS” \mathcal{C} on M .

b) The second condition for simplicity will be that \mathcal{C} is a (regular) GCS on M .

This is not so easy to formulate directly by means of (\mathcal{V}, g) , but the condition implies in particular that (\mathcal{V}, g) is nowhere transversally Riemannian.

Conversely, and this is the point, a nowhere transversally Riemannian lightlike manifold is locally simple: any point admits a simple neighbourhood.

Summarizing: there is a one to one correspondence between GCS structures and simple lightlike ones, the generic in one hand correspond to the generic in the other, and locally any nowhere transversally Riemannian lightlike metric gives rise to a GCS.

Example 2.1. *For the classical conformal sphere $(\mathbb{S}^n, \mathcal{C})$, the associated lightlike manifold \mathcal{V} is the lightcone Co^{n+1} seen as a lightlike submanifold in the Minkowski space \mathbb{R}^{n+2} endowed with the Lorentz form $q(x) = -x_1^2 + x_2^2 + \dots + x_{n+2}^2$. (Thus Co^{n+1} equals $\{x/q(x) = 0, x_1 > 0\}$).*

2.1.3. Sub-rigidity. A lightlike structure is an H -structure for H the orthogonal group of the standard lightlike scalar product $x_1^2 + \dots + x_{n-1}^2$ on \mathbb{R}^n . This structure has infinite type in Cartan's terminology, equivalently it is not rigid in Gromov's sense. We discussed in [3] sub-rigidity, a weaker property, that may be satisfied by lightlike metrics. *For $i < d$, a geometric structure is (d, i) **subrigid**, if any d -isometry which has a trivial i -jet at some point has in fact a trivial $i + 1$ -jet at that point.* In particular, $(d + 1, d)$ sub-rigidity coincides with usual d -rigidity.

2.1.4. Isometry groups. Let us call a *transvection* of (\mathcal{V}, g) any map $\mathcal{V} \rightarrow \mathcal{V}$ sending each leaf of \mathcal{N} to itself. A transvection is not necessarily isometric. In fact, any point admits in its neighbourhood a non-singular vector field generating (local) transvections, iff (\mathcal{V}, g) is transversally Riemannian.

If (\mathcal{V}, g) is simple, then we have a group morphism $\text{Iso}(\mathcal{V}, g) \rightarrow \text{Iso}(M, \mathcal{C})$. Its kernel is $\text{Iso}^{Tr}(\mathcal{V}, g)$, the group of isometric transvections. In the simple case, $\text{Iso}^{Tr}(\mathcal{V}, g)$ does not contain one parameter groups, but we can not conclude it is discrete, for instance because one does not know if $\text{Iso}(\mathcal{V}, g)$ is a Lie group?

Now, comparison between infinitesimal isometry groups of (\mathcal{V}, g) and (M, \mathcal{C}) is even more complicated. We can however, as stated in [3], relate sub-rigidity of (\mathcal{V}, g) to the rigidity of (M, \mathcal{C}) . Our second main result in the present article will be to provide a proof of $(3, 1)$ sub-rigidity of lightlike metrics based on Liouville Theorem for GCS:

Theorem 2.2. *In dimension ≥ 4 , a generic lightlike metric is $(3, 1)$ sub-rigid, that is a 3-isometry with trivial 1-jet has a trivial 2-jet.*

2.2. Remarks on other aspects. Many other natural questions can be asked about both local and global properties of GCS. For instance, one may try to weaken the genericity condition in Theorems 1.1 and 2.2, and also study global properties of isometric actions preserving GCS from the point of view of a global rigidity, say by asking a conjecture of Lichnerowicz type (see [10, 7, 9]). We will here briefly discuss the following other aspects:

2.2.1. *Pseudo-Riemannian case.* If one replaces Sym^+ by Sym^* , that is the space of non-degenerate quadratic forms (i.e. scalar pseudo-products) then one gets pseudo-Riemannian GCS, that are defined similarly by giving a curve in each $\text{Sym}^*(T_x M)$, for $x \in M$. All local facts extend to this wider framework.

2.2.2. *Anosov flows.* Let us give hints that Anosov flows always preserve GCS (of Riemannian type), although they never preserve classical Riemannian conformal structures (see for instance [11] for basic notions). Indeed, this will be a particular case of the general construction in 1.0.1. The point is that, one can choose the initial Riemannian metric g_0 so that the corresponding family $\phi_*^t g_0$ defines a (regular) GCS. Essentially, for any x , $t \rightarrow (\phi_*^t g_0)(x) \in \text{Sym}^+(T_x M)$ is a properly embedded curve \mathcal{C}_x , and thus $\mathcal{C} = \cup_x \mathcal{C}_x$ is a submanifold in $\text{Sym}^+(TM)$. To ensure this, one has to start with an adapted g_0 , that is, it is contracted on the stable bundle, and expanded on the unstable one.

Regarding genericity, let us make the following technical assumption (which seems that one can overcome). Denote by X the generating vector field of ϕ^t . Then assume that ϕ^t preserves a smooth supplementary sub-bundle $E \subset TM$, i.e. $TM = \mathbb{R}X \oplus E$ (such a E must be the sum of the stable and unstable bundles). Say E is defined by a 1 differential form η . Assume $g_0(X, X) = 1$, and consider now the GCS defined by $\phi_*^t g_0 + f(t)\eta \otimes \eta$, with $f(t)$ and $\frac{\partial f}{\partial t}$ positive for any t . This GCS is generic.

2.2.3. *A Geometric structure?* In general, GCS are neither H -structures (in Cartan's sense) nor geometric structures in the Gromov' sense (see [10, 7, 1, 6])! We already saw that a GCS \mathcal{C} is an H -structure iff it is infinitesimally homogeneous: all the curves $\mathcal{C}_x \subset \text{Sym}^+(T_x M)$ are linearly equivalent to a same curve $\mathcal{C}_0 \subset \text{Sym}^+(\mathbb{R}^n)$, when $x \in M$ (§1.0.3). Similarly, one can see that \mathcal{C} is a geometric structure in the Gromov' sense iff one can find a subset Z of curves in $\text{Sym}^+(\mathbb{R}^n)$, invariant under the $\text{GL}(n, \mathbb{R})$ -action, such Z is (naturally) a manifold (of finite dimension) and any curve \mathcal{C}_x is linearly equivalent to an element of Z . There are many technical difficulties with such a construction, but if one allows “cutting off” of the curves \mathcal{C}_x , i.e. restriction of their domains of definition, then, on a small open set U of M , one can find a manifold Z as above such that any $\mathcal{C}_x, x \in U$ is equivalent to an element of Z .

Such a construction, although local and non-universal, may be help at least from a formal point of view. It allows one to deal with GCS, for local questions (like rigidity) as if they were classical geometric structures.

2.2.4. *A-type?* In order to get a geometric structure of algebraic type (A-type), as defined in [10] (see also [7, 1]), one needs Z being an algebraic manifold and the $\text{GL}(n, \mathbb{R})$ -action on it algebraic (see [1]).

But, rigid geometric structures of algebraic type satisfy the Gromov's open dense orbit Theorem, that is if the isometry pseudo-group of the structure has a dense

orbit, then this one is open! In other words an open dense subset is locally homogeneous (see [10, 7, 4, 15]). However, one can see in the previous Anosov case that there are examples where such a local homogeneous subset can not exist. It then follows that despite one can manipulate in order to make GCS as geometric structures, it is not possible to do it within those of algebraic type!

3. SOME PRELIMINARIES

3.1. Case of H -structures. Let $H \subset \mathrm{GL}(n, \mathbb{R})$ be a closed subgroup and $\mathfrak{h} \subset \mathrm{End}(\mathbb{R}^n)$ its Lie algebra. Recall that the space \mathfrak{h}_d of d -prolongations is that of symmetric $(d+1)$ -multi-linear maps $A : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}^n$, such that for any given (u_1, \dots, u_d) , the endomorphism $u \rightarrow A(u, u_1, \dots, u_d)$ belongs to \mathfrak{h}

3.1.1. Algebraic structure.

Lemma 3.1. *Let \mathcal{C}_0 be a connected curve in $\mathrm{Sym}^+(\mathbb{R}^n)$ and H the connected component of its stabilizer in $\mathrm{GL}(n, \mathbb{R})$. Then H is semi-direct product $P \rtimes K$, where K is compact and acting trivially on \mathcal{C}_0 , and P is either trivial or a one parameter group acting transitively on \mathcal{C}_0 .*

Proof. \mathcal{C}_0 inherits from $\mathrm{Sym}^+(\mathbb{R}^n)$ a Riemannian metric (3.2.1) and becomes isometrics to an interval of \mathbb{R} . The case where \mathcal{C}_0 is a proper interval is somehow trivial, so let us consider the case where it is \mathbb{R} .

We have a representation $\rho : H \rightarrow \mathrm{Iso}(\mathbb{R})$. The kernel K of ρ is compact since it is a closed subgroup in the orthogonal group $O(b)$, for any $b \in \mathcal{C}_0$.

Since H is connected, $\rho(H)$ is either trivial or coincides with the translation group of \mathbb{R} . It then follows that if H is not compact, then $H/K \sim \mathbb{R}$. In this case, if P is a one parameter group not contained in K , then it projects bijectively on \mathbb{R} . Thus H is a semi-direct product $P \rtimes K$. □

3.1.2. Finiteness of type. Write $P = \exp tR$, and let \langle, \rangle be a scalar product preserved by K (as in the lemma above). An element of the Lie algebra of H has the form $C + \alpha R$, where C is antisymmetric ($C = -C^*$). It follows that a 2-prolongation A of H satisfies equation 1 in the generalized Braid Lemma 4.2 with $J = \langle, \rangle$ and $J'(\cdot, \cdot) = \langle (R + R^*)\cdot, \cdot \rangle$. It follows that H has type 2 when this form J' is non-degenerate. However, one can prove a more precise result. We have for instance the following:

Proposition 3.2. *Let $H = \{\exp tR \in \mathrm{GL}(n, \mathbb{R}), t \in \mathbb{R}\}$ be a one parameter group. Then an H -structure has finite type iff R has rank > 1 , in which case the type of H equals 1.*

Proof. The Lie algebra of H is $\mathfrak{h} = \{tR/t \in \mathbb{R}\}$.

An element of \mathfrak{h}_1 , i.e. a 1-prolongation is a symmetric bilinear mapping $L : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that for any x , the mapping $y \rightarrow L(x, y)$ belongs to \mathfrak{h} . It follows that L has the form $L(x, y) = t(x)Ry = \langle t, x \rangle Ry$ (where $\langle \cdot, \cdot \rangle$ is the usual Euclidean scalar product of \mathbb{R}^n). By symmetry of L , $\langle t, x \rangle Ry = \langle t, y \rangle Rx$, for all $x, y \in \mathbb{R}^n$, hence R has rank 1.

The converse is in fact true without assuming $\dim \mathfrak{h} = 1$: if \mathfrak{h} possesses an element of rank 1, then it has infinite type [12]. Indeed, if $Rx = \langle a, x \rangle v$ is such an element, then $L_d(x_1, \dots, x_{d+1}) = \langle a, x_1 \rangle \dots \langle a, x_{d+1} \rangle v$ determines a non-trivial d -prolongation. \square

3.2. Case of periodic curves.

3.2.1. Metric on Sym^+ . Let E be a vector space of dimension n . Its space of Euclidean Riemannian metrics (i.e. scalar products) $\text{Sym}^+(E)$ admits itself a canonical Riemannian (but no longer Euclidean) metric. To see it, observe first that $\text{Sym}^+(E)$ is an open set in $\text{Sym}(E)$, and hence the tangent space $T_b(\text{Sym}^+(E))$ at any point b can be identified with $\text{Sym}(E)$.

But a scalar product b defines a scalar product \bar{b} on $\text{Sym}(E)$: if (e_i) is a b -orthonormal basis, then $e_i^* \otimes e_j^*$ is a \bar{b} -orthonormal basis, where (e_i^*) is the dual basis (one has to check this does not depend on the basis).

Now, endow $T_b \text{Sym}^+(E)$ with \bar{b} .

Clearly, if F is another vector space, then any isomorphism $E \rightarrow F$ induces an isometry $\text{Sym}^+(E) \rightarrow \text{Sym}^+(F)$.

In fact, $\text{Sym}^+(E)$ is a symmetric space $\text{GL}(E)/\text{O}(b)$, where $\text{O}(b)$ is the orthogonal group of any $b \in \text{Sym}^+(E)$.

For, $E = \mathbb{R}$, one gets the metric $\frac{dx^2}{x^2}$ on \mathbb{R}^* .

For \mathbb{R}^2 , one gets the direct product $\mathbb{H}^2 \times \mathbb{R}$ (where \mathbb{H}^2 is the hyperbolic plane).

3.2.2. Topology.

Lemma 3.3. *Let \mathcal{C} be a GCS on M and assume that for some $x_0 \in M$, \mathcal{C}_{x_0} is a circle, i.e. a connected compact 1-manifold. Then, the same is true for nearby points. More precisely, there is a neighbourhood V of \mathcal{C}_{x_0} in \mathcal{C} and U a neighbourhood of x_0 such that $\pi : V \rightarrow U$ is a Seifert fibration.*

Proof. Let I be a small arc in M containing x_0 , then $S = \pi^{-1}(I)$ is a surface containing \mathcal{C}_{x_0} . Let S_0 be the connected component of \mathcal{C}_{x_0} in S . For I small enough, S_0 is a tubular neighbourhood of \mathcal{C}_{x_0} in S , and it is thus an annulus or a Moebius strip around \mathcal{C}_{x_0} . For the sake of simplicity of notation, let us limit ourselves to the annulus case. When x runs over I , the connected components of the \mathcal{C}_x in S_0 determine a 1-dimensional foliation \mathcal{F} of S_0 . But, each \mathcal{C}_x is closed in \mathcal{C} and hence each \mathcal{F} -leaf is closed. But such a foliation on the annulus is trivial, i.e. a trivial fibration on the interval (it is a Seifert fibration with monodromy $\mathbb{Z}/2\mathbb{Z}$ in the Moebius strip case).

Now, consider the same foliation \mathcal{F} , but on a neighbourhood $V = \pi^{-1}(U)$ in \mathcal{C} , where U is a small neighbourhood of x_0 in M . Since U can be generated by arcs, leaves of \mathcal{F} are all closed. But the holonomy of \mathcal{C}_{x_0} is trivial, since it is so above any interval. Hence the foliation is a fibration. \square

3.2.3. Geometry.

Proposition 3.4. *If \mathcal{C} has a circle fiber \mathcal{C}_{x_0} , then it is 1-rigid at x_0 . In fact, \mathcal{C} determines naturally a Riemannian metric near x_0 .*

Proof. \mathcal{C}_x is a circle in $\text{Sym}^+(T_x M)$. Consider an arc length parameterization $t \in [0, l] \rightarrow f(t) \in \mathcal{C}_x$, where l is the length of \mathcal{C}_x (f is defined up to a choice of an origin). The mean $\int f(t) dt$ is a canonically defined element of $\text{Sym}^+(T_x M)$, call it g_x . Since π is a smooth fibration, g_x depends smoothly on x , that is g is a smooth Riemannian metric defined on a neighbourhood of x_0 .

One then verifies that a d -isometry of \mathcal{C} is a d -isometry for g . By 1-rigidity of Riemannian metrics (that is a 2-isometry with trivial 1-jet has a trivial 2-jet) we deduce that \mathcal{C} is 1-rigid. \square

4. A GENERALIZED BRAID LEMMA

The classical (well known) Braid Lemma (see for instance [5]) states:

Lemma 4.1. *[Braid Lemma] If L is a trilinear map $E \times E \times E \rightarrow E$ on a vector space E , such that L is symmetric on the two first variables and skew-symmetric on the two last ones, then $L = 0$. In particular, if A is a bilinear map $E \times E \rightarrow E$ such that*

$$\langle A(U, V), W \rangle + \langle A(U, W), V \rangle = 0 \text{ for all } U, V \text{ and } W \text{ in } E,$$

where \langle, \rangle is an Euclidean scalar product, then $A = 0$.

If fact this is also true for pseudo-scalar products, that is for \langle, \rangle replaced by any non-degenerate symmetric bilinear form.

This statement is equivalent to the vanishing of 1-prolongations of the orthogonal group $O(E, \langle, \rangle)$, and thus to the 1-rigidity of a Riemannian structures.

We are going here to give a generalized Braid Lemma adapted to GC structures. Now, A will be a trilinear symmetric map $E \times E \times E \rightarrow E$, where E is a vector space which will be always assumed to have dimension ≥ 3 .

Proposition 4.2. *[Generalized Braid Lemma] Let A be a 3-linear vectorial form $E \times E \times E \rightarrow E$ satisfying:*

$$(1) \quad J(A(U, V, W), W') + J(A(U, V, W'), W) = K(U, V)J'(W, W')$$

where J, J' and $K: E \times E \rightarrow \mathbb{R}$ are some symmetric bilinear forms.

If J and J' are non-degenerate, then $A = 0$.

Proof. A direct computation gives us:

$$\begin{aligned} (2) \quad K(U, V)J'(W, W') &+ K(W, W')J'(U, V) \\ &= K(U, W)J'(V, W') + K(V, W')J'(U, W) \end{aligned}$$

(One just replaces each term as $K(U, V)J'(W, W')$ by its equivalent in the right hand of (1), and uses the fact that A is symmetric).

Now let W_1 and W_2 two J' -orthogonal vectors: $J'(W_1, W_2) = 0$. Let W_3 be a third vector J' -orthogonal to $\mathbb{R}W_1 + \mathbb{R}W_2$ and $J'(W_3, W_3) \neq 0$. Such W_3 exists because $\dim E \geq 3$ and J' is non-degenerate. We have:

$$\begin{aligned} K(W_1, W_2)J'(W_3, W_3) &+ K(W_3, W_3)J'(W_1, W_2) \\ &= K(W_1, W_3)J'(W_2, W_3) + K(W_2, W_3)J'(W_1, W_3) \end{aligned}$$

which implies $K(W_1, W_2) = 0$.

Write $K(U, V) = J'(U, P(V))$, where P is a J' -symmetric endomorphism of E .

Let W_1 with $J'(W_1, W_1) \neq 0$, and denote by W_1^\perp its J' -orthogonal. It follows that $P(W_1)$ is orthogonal to W_1^\perp , and hence $P(W_1) \in \mathbb{R}W_1$, that is W_1 is an eigenvector of P .

Thus P has all vectors W_1 with non-vanishing $J'(W_1, W_1)$ as eigenvectors. It follows that P is a homothety, that is $K = \alpha J'$ for some $\alpha \in \mathbb{R}$.

Now, using (2) for $V = U$, $W' = W$ and $J'(U, W) = 0$, we get:

$$\alpha J'(U, U)J'(W, W) = 0$$

which implies $\alpha = 0$ (since we can easily choose U and W with non-vanishing (square) J' -norm). Therefore, (1) becomes

$$J(A(U, V, W), W') + J(A(U, V, W'), W) = 0$$

which implies by the classical Braid Lemma that $A = 0$. □

5. PROOF OF THE GENERALIZED LIOUVILLE THEOREM

5.1. Position. Let (M, \mathcal{C}) be a GC manifold. The investigations in the present section are local in nature, so the manifold M can be identified with an open set in \mathbb{R}^n with coordinates (x^1, \dots, x^n) . In fact, we will work on a small neighbourhood of a fixed point p .

So far, we have studied the situation where a component of \mathcal{C}_p is a circle, and proved 1-rigidity in this case.

So we will now consider the opposite situation where all components of \mathcal{C}_p are injective images of \mathbb{R} . We choose one component and analyze \mathcal{C} around it. The projection π is not necessarily a locally trivial fibration, but restricting to a small neighbourhood of p (that we still denote M), any neighbourhood of a bounded arc of \mathcal{C}_p can be parameterized by a map

$$J : M \times I \rightarrow J(x, r) \in \text{Sym}^+(T_x M)$$

where I is a bounded interval of \mathbb{R} . We can also assume that $r \rightarrow J(x, r)$ is an arc length parameterization, for any x , although we do not need it. So locally,

$$J_{(x,r)} = \sum_{i,j} a_{ij}(x, r) dx^i dx^j$$

We will always assume \mathcal{C} nowhere transversally Riemannian, that is for some i, j , $\frac{\partial a_{ij}}{\partial r}(x, r) \neq 0$.

5.1.1. *Isometries.* For ϕ a diffeomorphism of M , we denote by ϕ'_x its derivative at x .

A diffeomorphism ϕ is isometric if its natural action on $\text{Sym}^+(TM)$ preserves \mathcal{C} , that is $\phi'_x(\mathcal{C}_x) = \mathcal{C}_{\phi(x)}$.

If the parameterization J were global, then the isometric property implies the existence of a re-parameterization $(x, r) \rightarrow k(x, r) \in \mathbb{R}$, such that:

$$(3) \quad J_{(\phi(x), k(x, r))}((\phi'_x)(U), (\phi'_x)(V)) - J_{(x, r)}(U, V) = 0, \forall U, V \text{ vector fields}$$

If the \mathcal{C} -curves are parameterized by length arc, then k has the form $k(x, r) = \delta(x) + r$.

Now, if the parameterization is not global, one just has to take care on the domains of definition, the same equation remains true. Precisely, one has a map $(x, r) \times M_1 \times I \rightarrow (\phi(x), k(x, r)) \in M_2 \times K$, where K is another interval, M_1 and M_2 are open subsets of M . However, for the sake of simplicity of notation, we will argue as if the parameterization is global.

5.1.2. *Infinitesimal isometries.* Assume now that $\phi(p) = p$. We defined in the introduction the fact that ϕ being a d -isometry at p by the condition that \mathcal{C} and its image $\phi^*(\mathcal{C})$ have a contact of order d along \mathcal{C}_p . As in the classical case, one shows this is equivalent to the usual vanishing condition up to order d , at p , of the previous equality. More precisely, for a given function k , and U, V vector fields, let

$$\Delta_k(U, V)(x, r) = J_{(\phi(x), k(x, r))}((\phi'_x)(U), (\phi'_x)(V)) - J_{(x, r)}(U, V)$$

Then, ϕ is a d -isometry, if there exists k such that the derivatives up to order $d-1$ of $\Delta_k(U, V)$ vanishes for any vector fields U and V .

Actually, it suffices to check this for U and V elements of a frame field on M , for example the natural vector fields $\frac{\partial}{\partial x^i}$. In the sequel, we will take U and V to be combination with constant coefficients of the $\frac{\partial}{\partial x^i}$.

5.1.3. *Notation.* The total derivative of a will be denoted a'_x , as used above for the diffeomorphism ϕ . Second and third total derivatives are denoted ϕ''_x and ϕ'''_x respectively.

For a function a on (x, r) we denote the derivative with respect to x at a point (p, r) by $(\mathbf{D}a)_{(p, r)}$ (i.e. the differential of $x \rightarrow a(x, r)$ where r is fixed). We similarly denote second derivative $(\mathbf{D}^2 a)_{(p, r)}$.

Regarding derivative with respect to r , we just denote it $(\partial_r a)(p, r)$.

As for derivatives of $(x, r) \rightarrow J(x, r)$, seen as a mapping valued in $\text{Sym}^+(\mathbb{R}^n)$, we set:

$$(J^{1,0})_{(x,r)}^W = \sum_{i,j} (\mathbf{D}a_{ij})_{(x,r)}(W) dx^i dx^j$$

and

$$J_{(x,r)}^{0,1} = \sum_{i,j} (\partial_r a_{ij})(x, r) dx^i dx^j.$$

and in the same way we put for all m and l in \mathbb{N} :

$$(J^{m,l})_{(x,r)}^{W_1, \dots, W_m} = \sum_{i,j} (\mathbf{D}^m \partial_r^l a_{ij})_{(x,r)}(W_1, W_2, \dots, W_m) dx^i dx^j$$

where $W, W_1, \dots, W_m \in T_x M$.

Finally, for $U \in T_x M$, we put $U_i = dx_i(U)$, $i = \overline{1, n}$.

5.1.4. Idea of proof of rigidity. We will consider a 3-isometry ϕ at a point p having a trivial 2-jet. Our objective will be to show that the third derivative $\phi_p''' = 0$. The goal of the next computations is to proof that $A = \phi_p'''$ satisfies our generalized Braid Lemma, with companion forms $J = J_{(p,r)}$, $K = (\mathbf{D}^2 k)_{(p,r)}$ and $J' = -J_{(p,r)}^{0,1}$.

5.2. 2-Isometries.

Lemma 5.1. *Assume ϕ to be isometric up to order 2 at p , and denote $(q, s) = (\phi(p), k(p, r))$. Then we have for all U and V in $T_p M$:*

$$(4) \quad J_{(p,r)}(U, V) = J_{(q,s)}(\phi_p'(U), \phi_p'(V)), \forall U, V \in T_p M$$

and furthermore

$$(5) \quad \begin{aligned} (J^{1,0})_{(p,r)}^W(U, V) &= (\mathbf{D}k)_{(p,r)}(W) J_{(q,s)}^{0,1}(\phi_p'(U), \phi_p'(V)) \\ &+ (J^{1,0})_{(q,s)}^{\phi_p'(W)}(\phi_p'(U), \phi_p'(V)) + J_{(q,s)}(\phi_p''(U, W), \phi_p'(V)) \\ &+ J_{(q,s)}(\phi_p'(U), \phi_p''(V, W)) \end{aligned}$$

Proof. Let us denote similarly as above $(y, t) = (\phi(x), k(x, r))$. We write equality at p of the first derivatives of the two sides of the equation

$$J_{(x,r)}(U, V) = J_{(\phi(x), k(x, r))}(\phi_x'(U), \phi_x'(V))$$

On one hand, we have:

$$\mathbf{D}[J_{x,r}(U, V)]_{(p,r)}(W) = \sum_{i,j} (\mathbf{D}a_{ij})_{(p,r)}(W) U_i V_j = (J^{1,0})_{(p,r)}^W(U, V)$$

On the other hand, set

$$b_{ij}(x, r) = a_{ij} \circ \psi(x, r) \text{ where } \psi(x, r) = (\phi(x), k(x, r))$$

then

$$\begin{aligned} (\mathbf{D}b_{ij})_{(x,r)}(W) &= (a_{ij})'_{\psi(x,r)} \circ (\mathbf{D}\psi)_{(x,r)}(W) = (a_{ij})'_{(y,t)}(\phi_x'(W), (\mathbf{D}k)_{(x,r)}(W)) \\ &= (\mathbf{D}a_{ij})_{(y,t)}(\phi_x'(W)) + (\partial_r a_{ij})(y, t)(\mathbf{D}k)_{(x,r)}(W) \end{aligned}$$

Now the derivative with respect to x of

$$(x, r) \rightarrow J_{(\phi(x), k(x, r))}(\phi'_x(U), \phi'_x(V)) = \sum_{i, j} b_{ij}(x, r)(\phi'_x(U))_i(\phi'_x(V))_j$$

equals:

$$\begin{aligned} & \sum_{i, j} (\mathbf{D}b_{ij})_{(x, r)}(W)(\phi'_x(U))_i(\phi'_x(V))_j + b_{ij}(x, r)\mathbf{D}[(\phi'_x(U))_i(\phi'_x(V))_j]_{(x, r)}(W) \\ &= \sum_{i, j} (\mathbf{D}a_{ij})_{(y, t)}(\phi'_x(W))(\phi'_x(U))_i(\phi'_x(V))_j \\ & \quad + \sum_{i, j} (\mathbf{D}k)_{(x, r)}(W)(\partial_r a_{ij})(y, t)(\phi'_x(U))_i(\phi'_x(V))_j \\ & \quad + \sum_{i, j} b_{ij}(x, r)(\phi''_x(U, W))_i(\phi'_x(V))_j + \sum_{i, j} b_{ij}(x, r)(\phi'_x(U))_i(\phi''_x(V, W))_j \\ &= (J^{1,0}_{(y, t)}(\phi'_x(W))(\phi'_x(U), \phi'_x(V)) + (\mathbf{D}k)_{(x, r)}(W)J^{0,1}_{(y, t)}(\phi'_x(U), \phi'_x(V)) \\ & \quad + J_{(y, t)}(\phi''_x(U, W), \phi'_x(V)) + J_{(y, t)}(\phi'_x(U), \phi''_x(V, W))) \end{aligned}$$

replacing x by p we get (5). \square

Corollary 5.2. 1) If ϕ is a 1-isometry at p , with a trivial 1-jet (at p), then

$$(6) \quad k(p, r) = r \text{ for any } r$$

2) If ϕ is a 2-isometry at p with a trivial 2-jet, then

$$(\mathbf{D}k)_{(p, r)} = 0 \text{ for any } r$$

Proof. 1) Since ϕ is a 1-isometry then by (4) we have for all $U, V \in T_p M$

$$J_{(p, r)}(U, V) = J_{(\phi(p), k(p, r))}(\phi'_p(U), \phi'_p(V))$$

But $\phi(p) = p$ and $\phi'_p = Id$ and so

$$J_{(p, r)}(U, V) = J_{(p, k(p, r))}(U, V), \forall U, V \in T_p M$$

Remember however that by our hypotheses (in the beginning of the present §), \mathcal{C}_p is a 1-dimensional submanifold without compact components, in particular, $r \rightarrow J(p, r)$ is injective, and hence $k(p, r) = r$.

2) If ϕ is a 2-isometry at p with a trivial 2-jet, then we have furthermore $\phi''_p = 0$. Putting $\phi'_p = Id$, $\phi(p) = p$ and $k(p, r) = r$, we get:

$$(\mathbf{D}k)_{(p, r)}(W)J^{0,1}_{p, r}(U, V) = 0 \text{ for any } W, U, V$$

But, by definition of GCS, the curve $r \rightarrow J(p, r) \in \text{Sym}^+(T_p M)$ is non-singular, and hence $(U, V) \rightarrow J^{0,1}_{p, r}(U, V)$ is a non-vanishing bilinear form, and so $\mathbf{D}k_{(p, r)} = 0$ as claimed. \square

5.3. 3-Isometries. The following lemma will be the last step towards application of the generalized Braid Lemma to our situation.

Lemma 5.3. *If ϕ is a 3-isometry at p with a trivial 2-jet ($\phi(p) = p$, $\phi'_p = Id$ and $\phi''_p = 0$), then for all U, V, W_1 and W_2 in $T_p M$*

$$(7) \quad \begin{aligned} J_{(p,r)}(\phi'''_p(U, W_1, W_2), V) &+ J_{(p,r)}(U, \phi'''_p(V, W_1, W_2)) \\ &= -(\mathbf{D}^2 k)_{(p,r)}(W_1, W_2)(J^{0,1})_{(p,r)}(U, V) \end{aligned}$$

Proof. The hypothesis means equality of the second derivatives with respect to x , at p , of both sides of the equation:

$$(8) \quad J_{(x,r)}(U, V)(W_1, W_2) = J_{(\phi(x), k(x,r))}(\phi'_x(U), \phi'_x(V))(W_1, W_2)$$

for all $U, V, W_1, W_2 \in T_p M$.

For the left side, this derivative is nothing but

$$(J^{2,0})_{(p,r)}^{(W_1, W_2)}(U, V)$$

Let us now consider the right side. Keeping, the same notations b_{ij} and ψ as in the previous proof, the second derivative with respect to x equals:

$$\begin{aligned} &\sum_{i,j} \mathbf{D}^2[b_{ij}(x,r)(\phi'_x(U))_i(\phi'_x(V))_j]_{(x,r)}(W_1, W_2) \\ &= \sum_{i,j} (\mathbf{D}^2 b_{ij})_{(x,r)}(W_1, W_2)(\phi'_x(U))_i(\phi'_x(V))_j \\ &+ \sum_{i,j} (\mathbf{D} b_{ij})_{(x,r)}(W_1)(\phi''_x(U, W_2))_i(\phi'_x(V))_j + \sum_{i,j} (\mathbf{D} b_{ij})_{(x,r)}(W_1)(\phi'_x(U))_i(\phi''_x(V, W_2))_j \\ &+ \sum_{i,j} (\mathbf{D} b_{ij})_{(x,r)}(W_2)(\phi''_x(U, W_1))_i(\phi'_x(V))_j + \sum_{i,j} (\mathbf{D} b_{ij})_{(x,r)}(W_2)(\phi'_x(U))_i(\phi''_x(V, W_1))_j \\ &+ \sum_{i,j} b_{ij}(x,r)(\phi''_x(U, W_1))_i(\phi''_x(V, W_2))_j + \sum_{i,j} b_{ij}(x,r)(\phi''_x(U, W_2))_i(\phi''_x(V, W_1))_j \\ &+ \sum_{i,j} b_{ij}(x,r)(\phi'''_x(U, W_1, W_2))_i(\phi'_x(V))_j + \sum_{i,j} b_{ij}(x,r)(\phi'_x(U))_i(\phi'''_x(V, W_1, W_2))_j \\ &= B_1(x,r) + B_2(x,r) + B_3(x,r) + B_4(x,r) + B_5(x,r) + B_6(x,r) + B_7(x,r) \\ &\quad + B_8(x,r) + B_9(x,r) \end{aligned}$$

All terms which contain ϕ'' vanish at $x = p$ since ϕ has a trivial 2-jet at p , and so:

$$B_2(p,r) = B_3(p,r) = B_4(p,r) = B_5(p,r) = B_6(p,r) = B_7(p,r) = 0$$

Again by this triviality and (6), we have $k(p,r) = r$, and hence,

$$B_8(p,r) = J_{(p,r)}(\phi'''_p(U, W_1, W_2), V) \text{ and } B_9(p,r) = J_{(p,r)}(U, \phi'''_p(V, W_1, W_2))$$

It remains to calculate B_1 , for which we have:

$$\begin{aligned}
(\mathbf{D}^2 b_{ij})_{(x,r)}(W_1, W_2) &= (\mathbf{D}^2(a_{ij} \circ \psi))_{(x,r)}(W_1, W_2) \\
&= (a_{ij})''_{\psi(x)}((\mathbf{D}\psi)_{(x,r)}(W_1), (\mathbf{D}\psi)_{(x,r)}(W_2)) \\
&\quad + (a_{ij})'_{\psi(x)}((\mathbf{D}^2\psi)_{(x,r)}(W_1, W_2)) \\
&= (a_{ij})''_{\psi(x)}((\phi'_x(W_1), (\mathbf{D}k)_{(x,r)}(W_1)), (\phi'_x(W_2), (\mathbf{D}k)_{(x,r)}(W_2))) \\
&\quad + (a_{ij})'_{\psi(x)}(\phi''_x(W_1, W_2), (\mathbf{D}^2k)_{(x,r)}(W_1, W_2)) \\
&= B_{11}(x, r) + B_{12}(x, r)
\end{aligned}$$

since $k(p, r) = r$, $(\mathbf{D}k)_{(p,r)} = 0$, $\phi'_p = Id$ and $\phi''_p = 0$, we get

$$B_{11}(p, r) = (\mathbf{D}^2 a_{ij})_{(p,r)}(W_1, W_2)$$

and

$$B_{12}(p, r) = (\partial_r a_{ij})(p, r)(\mathbf{D}^2 k)_{(p,r)}(W_1, W_2)$$

and hence

$$\begin{aligned}
B_1(p, r) &= \sum_{i,j} (\mathbf{D}^2 b_{ij})_{(p,r)}(W_1, W_2)(U)_i(V)_j \\
&= \sum_{i,j} (\mathbf{D}^2 a_{ij})_{(p,r)}(W_1, W_2)(U)_i(V)_j + (\mathbf{D}^2 k)_{(p,r)}(W_1, W_2) \sum_{i,j} (\partial_r a_{ij})(p, r)(U)_i(V)_j \\
&= (J^{2,0})_{(p,r)}^{W_1, W_2}(U, V) + (\mathbf{D}^2 k)_{(p,r)}(W_1, W_2) J_{(p,r)}^{0,1}(U, V)
\end{aligned}$$

Summarizing, equality of second derivatives of (8) at p gives

$$\begin{aligned}
(J^{2,0})_{(p,r)}^{(W_1, W_2)}(U, V) &= (J^{2,0})_{(p,r)}^{(W_1, W_2)}(U, V) + (\mathbf{D}^2 k)_{(p,r)}(W_1, W_2)(J^{0,1})_{(p,r)}(U, V) \\
&\quad + J_{(p,r)}(\phi_p'''(U, W_1, W_2), V) + J_{(p,r)}(U, \phi_p'''(V, W_1, W_2))
\end{aligned}$$

Equivalently

$$\begin{aligned}
J_{(p,r)}(\phi_p'''(U, W_1, W_2), V) &+ J_{(p,r)}(U, \phi_p'''(V, W_1, W_2)) \\
&= -(\mathbf{D}^2 k)_{(p,r)}(W_1, W_2) J_{(p,r)}^{0,1}(U, V)
\end{aligned}$$

□

5.4. End of the proof of Theorem 1.1. If ϕ is a 3-isometry at p with a trivial 2-jet, then by (7) and the generalized Braid Lemma applies with $A = \phi_p'''$, $J = J_{(p,r)}$, $K = (\mathbf{D}^2 k)_{(p,r)}$ and $J' = -J_{(p,r)}^{0,1}$, one conclude that $\phi_p''' = 0$, which is exactly the meaning of rigidity at order 2. □

6. SUB-RIGIDITY OF LIGHTLIKE METRICS, PROOF OF THEOREM 2.2

6.1. Position of the problem. Let (\mathcal{V}, g) be a lightlike n -dimensional manifold. Since we are dealing with questions local in nature, so we can assume \mathcal{V} is a small chart domain, say $\mathcal{V} = M \times I$ where I is an interval. The factor I corresponds to the characteristic foliation tangent to the kernel of g .

In an adapted coordinate system $(x_1, x_2, \dots, x_{n-1}, t)$ (t corresponds to I), the lightlike metric takes the form

$$g_{(x,t)} = \sum_{i,j} a_{ij}(x,t) dx^i dx^j$$

This gives for any fixed r , a Riemannian metric on $M \times \{r\}$. By endowing $T_x M$ with the scalar products $g_{(x,r)}$, $r \in I$, we get a GCS on M , once we assume g nowhere transversally Riemannian, that is $\frac{\partial}{\partial t} g_{(x,t)} \neq 0$ (see 2.1). Recall that g is said to be generic if $\frac{\partial}{\partial t} g_{(x,t)} = \sum_{i,j} \frac{\partial a_{ij}}{\partial t}(x,t) dx^i dx^j$ is non-degenerate.

A diffeomorphism Ψ of M has the form $\Psi = (\phi, \delta)$ where $\phi : M \times I \rightarrow M$ and $\delta : M \times I \rightarrow I$.

If Ψ is isometric, then it preserves the I foliation, and hence ϕ does not depend of t . Furthermore, for any U and V in $T_{(x,t)} \mathcal{V}$

$$(9) \quad g_{(x,t)}(U, V) = g_{\Psi(x,t)}(\Psi'_{(x,t)}(U), \Psi'_{(x,t)}(V))$$

A tangent vector $U \in T_{(x,t)} \mathcal{V}$ will be denoted $(U_M, U_I) \in T_x M \times T_t I$.

6.2. Step 1: a partial 1-rigidity. *If Ψ is isometric up to order 2, with a trivial 1-jet at a point $(p, r) \in M$ then $\phi''_{(p,r)} = 0$.*

Proof. If $\Psi = (\phi, \delta)$ is isometric up to order 2 then the equality of (9) holds for the derivatives at $(x, t) = (p, r)$. We have

$$(10) \quad (g_{(x,t)}(U, V))'_{(p,r)}(W) = \sum_{i,j} (a_{ij})'_{(p,r)}(W) U_i V_j$$

In the other hand, if we denote a generic point (x, t) by v

$$g_{\Psi(v)}(\Psi'_v(U), \Psi'_v(V)) = \sum_{i,j} a_{ij}(\Psi(v)) (\Psi'_v(U))_i (\Psi'_v(V))_j$$

a derivation gives

$$(11) \quad \sum_{i,j} (a_{ij})'_{\Psi(v)}(\Psi'_v(W)) (\Psi'_v(U))_i (\Psi'_v(V))_j + \sum_{i,j} a_{ij}(\Psi(v)) (\Psi''_v(U, W))_i (\Psi'_v(V))_j \\ + \sum_{i,j} a_{ij}(\Psi(v)) (\Psi'_v(U))_i (\Psi''_v(V, W))_j$$

We have

$$(\Psi'_v(U))_i = (\phi'_v(U))_i \text{ and } (\Psi''_v(U))_i = (\phi''_v(U))_i$$

using the triviality of the 1-jet of Ψ we get

$$\Psi(p, r) = (p, r), \Psi'_{(p,r)} = Id$$

then (11) becomes

$$\sum_{i,j} (a_{ij})'_{(p,r)}(W) U_i V_j + \sum_{i,j} a_{ij}(p, r) (\phi''_{(p,r)}(U, W))_i (V)_j + \sum_{i,j} a_{ij}(p, r) (U)_i (\phi''_{(p,r)}(V))_j$$

Therefore, the equality with (10) gives

$$\sum_{i,j} a_{ij}(p,r)(\phi''_{(p,r)}(U,W))_i(V)_j + \sum_{i,j} a_{ij}(p,r)(U)_i(\phi''_{(p,r)}(V,W))_j = 0$$

that is

$$g_{(p,r)}(\phi''_{(p,r)}(U,W), V) + g_{(p,r)}(U, \phi''_{(p,r)}(V,W)) = 0$$

By the Braid Lemma we conclude that $\phi''_{(p,r)} = 0$. \square

6.3. Step 2: the ϕ -part. Assume the lightlike structure generic. If $\Psi = (\phi, \delta)$ is a 3-isometry at (p, r) with a trivial 1-jet, then $\phi'''_{(p,r)} = 0$.

Proof. If Ψ was a true isometry, then it acts, via ϕ , on M seen as the quotient space of the characteristic foliation (in particular it does not depend on r), and it preserves the GCS on it. The genericity hypothesis allows one to apply Theorem 1.1 to conclude that $\phi'''_{(p,r)} = 0$.

Now, we want to apply the same argument when Ψ is merely isometric up to order 3 at (p, r) (and has a trivial 1-jet). The idea then is to show that the diffeomorphism $x \rightarrow \varphi(x) = \phi(x, r)$ is a 3-isometry of the GCS of M . The expected k -shift of φ (see 5.1.1) will be nothing but δ . In other words, we want φ to satisfy the following equation up to order 3 at p :

$$(12) \quad g_{(x,r)}(U, V) = g_{(\varphi(x), \delta(x,r))}(\varphi'_x(U), \varphi'_x(V))$$

This property of φ , follows from the similar one of Φ , that is, it satisfies (9) up to order 3, and remembering that $\phi''_{(p,r)} = 0$ by the previous step. Indeed, let us derive twice the equation satisfied by Ψ at $\nu = (p, r)$:

$$g_{(x,t)}(U, V) = g_{\Psi(x,t)}(\Psi'_{(x,t)}(U), \Psi'_{(x,t)}(V))$$

we get

$$(13) \quad \begin{aligned} g_{\nu}^{W_1, W_2}(U, V) &= g_{\Psi(\nu)}^{\Psi''_{\nu}(W_1, W_2)}(\Psi'_{\nu}(U), \Psi'_{\nu}(V)) + g_{\Psi(\nu)}^{\Psi'_{\nu}(W_1), \Psi'_{\nu}(W_2)}(\Psi'_{\nu}(U), \Psi'_{\nu}(V)) \\ &+ g_{\Psi(\nu)}^{\Psi'_{\nu}(W_1)}(\Psi''_{\nu}(U, W_2), \Psi'_{\nu}(V)) + g_{\Psi(\nu)}^{\Psi'_{\nu}(W_1)}(\Psi'_{\nu}(U), \Psi''_{\nu}(V, W_2)) \\ &+ g_{\Psi(\nu)}^{\Psi'_{\nu}(W_2)}(\Psi''_{\nu}(U, W_1), \Psi'_{\nu}(V)) + g_{\Psi(\nu)}^{\Psi'_{\nu}(W_2)}(\Psi'_{\nu}(U), \Psi''_{\nu}(V, W_1)) \\ &+ g_{\Psi(\nu)}(\Psi''_{\nu}(U, W_1), \Psi''_{\nu}(V, W_2)) + g_{\Psi(\nu)}(\Psi''_{\nu}(U, W_2), \Psi''_{\nu}(V, W_1)) \\ &+ g_{\Psi(\nu)}(\Psi'''_{\nu}(U, W_1, W_2), \Psi'_{\nu}(V)) + g_{\Psi(\nu)}(\Psi'_{\nu}(U), \Psi'''_{\nu}(V, W_1, W_2)) \end{aligned}$$

where as before

$$g_v^W = \sum_{i,j} (a_{ij})'_v(W) dx^i dx^j$$

and

$$g_v^{W_1, W_2} = \sum_{i,j} (a_{ij})''_v(W_1, W_2) dx^i dx^j$$

But

$$g_v(U, V) = g_v(U, V_M) = g_v(U_M, V) = g_v(U_M, V_M),$$

and the same thing for g_v^W and $g_v^{W_1, W_2}$, then (13) becomes

$$\begin{aligned}
 (14) \quad g_\nu^{W_1, W_2}(U, V) &= g_{\Psi(\nu)}^{\Psi'_\nu(W_1, W_2)}(\Psi'_\nu(U), \Psi'_\nu(V)) + g_{\Psi(\nu)}^{\Psi'_\nu(W_1), \Psi'_\nu(W_2)}(\Psi'_\nu(U), \Psi'_\nu(V)) \\
 &\quad + g_{\Psi(\nu)}^{\Psi'_\nu(W_1)}(\phi''_\nu(U, W_2), \Psi'_\nu(V)) + g_{\Psi(\nu)}^{\Psi'_\nu(W_1)}(\Psi'_\nu(U), \phi''_\nu(V, W_2)) \\
 &\quad + g_{\Psi(\nu)}^{\Psi'_\nu(W_2)}(\phi''_\nu(U, W_1), \Psi'_\nu(V)) + g_{\Psi(\nu)}^{\Psi'_\nu(W_2)}(\Psi'_\nu(U), \phi''_\nu(V, W_1)) \\
 &\quad + g_{\Psi(\nu)}(\phi''_\nu(U, W_1), \phi''_\nu(V, W_2)) + g_{\Psi(\nu)}(\phi''_\nu(U, W_2), \phi''_\nu(V, W_1)) \\
 &\quad + g_{\Psi(\nu)}(\phi'''_\nu(U, W_1, W_2), \Psi'_\nu(V)) + g_{\Psi(\nu)}(\Psi'_\nu(U), \phi'''_\nu(V, W_1, W_2))
 \end{aligned}$$

Since Ψ has a trivial 1-jet at ν , $\Psi(\nu) = (p, r)$, $\Psi'_\nu = Id$ and $\phi''_\nu = 0$ (Step 1), and so

$$\begin{aligned}
 (15) \quad g_{(p,r)}(\phi'''_{(p,r)}(U, W_1, W_2), V) &+ g_{(p,r)}(U, \phi'''_{(p,r)}(V, W_1, W_2)) \\
 &= -g_{(p,r)}^{\Psi''_{(p,r)}(W_1, W_2)}(U, V)
 \end{aligned}$$

But

$$\begin{aligned}
 g_{(p,r)}^{\Psi''_{(p,r)}(W_1, W_2)}(U, V) &= \sum_{i,j} (a_{ij})'_{(p,r)}(\Psi''_{(p,r)}(W_1, W_2))(U)_i (V)_j \\
 &= \sum_{i,j} (\mathbf{D}a_{ij})_{(p,r)}(\phi''_{(p,r)}(W_1, W_2)) + \delta''_{(p,r)}(W_1, W_2)(\partial_t a_{ij})(p, r)(U)_i (V)_j \\
 &= \sum_{i,j} \delta''_{(p,r)}(W_1, W_2)(\partial_t a_{ij})(p, r)(U)_i (V)_j \\
 &= \delta''_{(p,r)}(W_1, W_2)g_{(p,r)}^{0,1}(U, V)
 \end{aligned}$$

Thus (15) gives

$$\begin{aligned}
 (16) \quad g_{(p,r)}(\phi'''_{(p,r)}(U, W_1, W_2), V) &+ g_{(p,r)}(U, \phi'''_{(p,r)}(V, W_1, W_2)) \\
 &= -\delta''_{(p,r)}(W_1, W_2)g_{(p,r)}^{0,1}(U, V)
 \end{aligned}$$

Finally, apply the Generalized Braid Lemma to $J = g_{(p,r)}$, $K = \delta''_{(p,r)}$, $J' = -g_{(p,r)}^{0,1}$ and $A = \phi'''_{(p,r)}$ to get $\phi'''_{(p,r)} = 0$. \square

6.4. Step 3: the δ -part, end of proof of Theorem 2.2. *Let M be a generic lightlike manifold. If $\Psi = (\phi, \delta)$ is a 3-isometry at (p, r) with a trivial 1-jet at (p, r) then Ψ has a trivial 2-jet at (p, r) .*

Proof. By step 2 we have $\phi'''_{(p,r)} = 0$, so (16) gives

$$\delta''_{(p,r)}(W_1, W_2) \sum_{i,j} (\partial_t a_{ij})(p, r) U_i V_j = 0$$

Remember that g is nowhere transversally Riemannian, thus $\delta''_{(p,r)} = 0$, and hence Ψ has a trivial 2-jet.

Theorem 2.2, that is (3,1)-sub-rigidity of generic lightlike metrics follows. \square

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